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Scattering of Plane Waves in Self-Dual Yang-Mills Theory

Vladimir E. Korepin[1–3] * and Takeshi Oota[3] †

[1]*Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794–3840, U. S. A.*

[2]*Sankt Petersburg Department of Mathematical Institute of Academy of Sciences of Russia*

[3]*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan*

ABSTRACT

We consider the classical self-dual Yang-Mills equation in 3+1-dimensional Minkowski space. We have found an exact solution, which describes scattering of n plane waves. In order to write the solution in a compact form, it is convenient to introduce a scattering operator $\hat{T}$. It acts in the direct product of three linear spaces: 1) universal enveloping of $su(N)$ Lie algebra, 2) n-dimensional vector space and 3) space of functions defined on the unit interval.

*e-mail: `korepin@insti.physics.sunysb.edu`

†e-mail: `toota@yukawa.kyoto-u.ac.jp`

1 Introduction

We consider the classical Yang-Mills field with value in $su(N)$ algebra in 3+1-dimensional Minkowski space. We study the self-dual equation:

$$F_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{1.1}$$

Study of the self-dual Yang-Mills equation is important for understanding of QCD [1]–[7].

We fix the gauge following [9]:

$$A_{0-z} = 0, \quad A_{x+iy} = 0, \quad A_{0+z} = \sqrt{2}\partial_{x+iy}\Phi, \quad A_{x-iy} = \sqrt{2}\partial_{0-z}\Phi. \tag{1.2}$$

Here $A_{0\pm z} = A_0 \pm A_z$, $A_{x\pm iy} = A_x \mp iA_y$.

The self-dual Yang-Mills equation for scalar $su(N)$ field Φ looks as follows

$$\Box\Phi - ig[\partial_{x+iy}\Phi, \partial_{0-z}\Phi] = 0. \tag{1.3}$$

It is associated with a cubic action [10]. Following [8] we start looking for the solution of eq.(1.3) using perturbation theory in coupling constant g

$$\Phi(x) = \sum_{m=1}^{\infty}\Phi^{(m)}(x). \tag{1.4}$$

Here $\Phi^{(m)}$ depends on the coupling as g^{m-1}. The first term satisfies a linear equation

$$\Box\Phi^{(1)} = 0. \tag{1.5}$$

We choose $\Phi^{(1)}$ as a sum of n plane waves

$$\Phi^{(1)}(x) = -i\sum_{j=1}^{n}T^{a_j}e^{-ik_jx}f(k_j). \tag{1.6}$$

Here T^a are $su(N)$ generators

$$\begin{aligned} [T^a, T^b] &= i\sqrt{2}f^{abc}T^c, \\ trT^aT^b &= \delta^{ab}. \end{aligned} \tag{1.7}$$

All k_j are n different light-cone vectors $k_j^2 = 0$ and $f(k)$ is a function with support on the light-cone. We will also use the following notations:

$$Q_j = \frac{(k_j)_{0+z}}{(k_j)_{x+iy}} = \frac{(k_j)_{x-iy}}{(k_j)_{0-z}}. \tag{1.8}$$

We have found explicit expression for $\Phi^{(m)}$. The first two terms coincide with [8], all other $\Phi^{(m)}$ ($m \geq 3$) are different.

Let us explain our solution.

We shall use an abbreviation:

$$\phi(j) = T^{a_j} e^{-ik_j x} f(k_j). \tag{1.9}$$

We introduce the following function:

$$V(a) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} a^n = \oint \frac{dt}{2\pi i} \frac{e^{1/t+at}}{t} = I_0(2\sqrt{a}). \tag{1.10}$$

Here I_0 is the modified Bessel function of the first kind. The integration contour is a circle around zero. We integrate in positive direction.

Let us define a linear operator:

$$\begin{aligned} & T(\alpha_1, \alpha_2; j_1, j_2) \\ = \; & g\phi(j_1)P(j_1, j_2) \int_0^{\infty} ds \; e^{-s} V\Big(s\alpha_1 g\phi(j_1)P(j_1, j_2)\Big) V\Big(s\alpha_2 g\phi(j_2)P(j_1, j_2)\Big). \end{aligned} \tag{1.11}$$

Here j_1 and j_2 run through n values. Integration variables $\alpha_{1,2}$ belong to a unit interval $[0, 1]$. We shall consider T as an operator in direct product of n-dimensional vector space and as an integral operator. The kernel $T(\alpha_1, \alpha_2; j_1, j_2)$ takes values in an universal enveloping algebra of $su(N)$. We are using $P(j_1, j_2)$ which is defined by

$$P(j_1, j_2) = \begin{cases} (Q_{j_1} - Q_{j_2})^{-1} & \text{for } \; j_1 \neq j_2 \\ \\ 0 & \text{for } \; j_1 = j_2. \end{cases} \tag{1.12}$$

The kernel $T(\alpha, \alpha'; j, j')$ depends only on the j-th and j'-th plane waves. It vanishes if $j = j'$.

The function (1.11) is a kernel of an operator $\hat{T}$

$$(\hat{T})_{(\alpha_1; j_1), (\alpha_2; j_2)} = T(\alpha_1, \alpha_2; j_1, j_2),$$

whose index $(\alpha; j)$ takes a value in $[0,1] \times \{1, 2, \ldots, n\}$. It acts on a "vector" $(\vec{f})_{(\alpha;j)}$ which takes a value in the universal enveloping algebra as follows:

$$(\hat{T}\vec{f})_{(\alpha;j)} = \sum_{j'=1}^{n} \int_0^1 d\alpha' T(\alpha, \alpha'; j, j')(\vec{f})_{(\alpha';j')}. \tag{1.13}$$

We call $\hat{T}$ the 'scattering operator'.

We introduce two special "vector"s (see (1.9))

$$(\vec{\phi})_{(\alpha;j)} = \phi(j), \quad (\vec{\phi}_0)_{(\alpha;j)} = 1. \tag{1.14}$$

For example, a scalar product of $\vec{\phi}_0$ and an arbitrary vector function $\vec{f}$ is equal to:

$$\vec{\phi}_0 \cdot \vec{f} = \sum_{j=1}^{n} \int_0^1 d\alpha \; (f)_{(\alpha;j)}.$$

Now all the notations are prepared in order to write down the solution of the self-dual equation (1.3) which we have found:

$$\Phi(x) = -i\vec{\phi}_0 \cdot \left(\frac{1}{1 - \hat{T}} \right) \vec{\phi}. \tag{1.15}$$

This is the main result of our paper.

2 Solving recursion relations

Substituting eq.(1.4) into the self-dual equation (1.3) gives the following recursion relations for $\Phi^{(m)}(x)$:

$$\Box \Phi^{(m)}(x) = ig \sum_{j=1}^{m-1} (\partial_{x+iy}\Phi^{(j)}\partial_{0-z}\Phi^{(m-j)} - \partial_{0-z}\Phi^{(j)}\partial_{x+iy}\Phi^{(m-j)}). \tag{2.1}$$

It is convenient to introduce the following object:

$$X(1,2) = (k_1)_{x+iy}(k_2)_{0-z} - (k_1)_{0-z}(k_2)_{x+iy}. \tag{2.2}$$

It has the property:

$$2k_1 \cdot k_2 = X(1,2)(Q_1 - Q_2). \tag{2.3}$$

The lowest component of $\Phi(x)$ in (1.4) satisfies the free equation:

$$\Box \Phi^{(1)} = 0. \tag{2.4}$$

A solution is given by

$$\Phi^{(1)}(x) = -i\sum_{j=1}^{n} T^{a_j} e^{-ik_j x} f(k_j), \qquad k_j^2 = 0. \tag{2.5}$$

We shall use the notation:

$$\phi(j) = T^{a_j} e^{-ik_j x} f(k_j). \tag{2.6}$$

Let us analyze $\Phi^{(2)}(x)$. The recursion relation (2.1) is

$$\begin{aligned}
\Box\Phi^{(2)}(x) &= ig(\partial_{x+iy}\Phi^{(1)}\partial_{0-z}\Phi^{(1)} - \partial_{0-z}\Phi^{(1)}\partial_{x+iy}\Phi^{(1)}) \\
&= ig\sum_{j_1}\sum_{j_2}(\partial_{x+iy}\phi(j_1)\partial_{0-z}\phi(j_2) - \partial_{0-z}\phi(j_1)\partial_{x+iy}\phi(j_2)) \\
&= -ig\sum_{j_1}\sum_{j_2}\phi(j_1)\phi(j_2)X(j_1,j_2) \\
&= ig\sum_{j_1\neq j_2}\phi(j_1)\phi(j_2)X(j_1,j_2).
\end{aligned} \tag{2.7}$$

Here we have used the property $X(j_1, j_1) = 0$.

Equation (2.7) can be solved by:

$$\begin{aligned}
\Phi^{(2)}(x) &= -ig\sum_{j_1}\sum_{j_2(\neq j_1)}\phi(j_1)\phi(j_2)X(j_1,j_2)/(k_{j_1}+k_{j_2})^2 \\
&= -ig\sum_{j_1}\sum_{j_2(\neq j_1)}\phi(j_1)\phi(j_2)(Q_{j_1}-Q_{j_2})^{-1}.
\end{aligned} \tag{2.8}$$

In general, we can add any terms which satisfy the free equation to this solution $\Phi^{(2)}(x)$. We choose the minimal solution (2.8).

Next, we consider an equation for $\Phi^{(3)}(x)$.

$$\begin{aligned}
\Box\Phi^{(3)}(x) &= ig(\partial_{x+iy}\Phi^{(1)}\partial_{0-z}\Phi^{(2)} - \partial_{0-z}\Phi^{(1)}\partial_{x+iy}\Phi^{(2)}) \\
&+ ig(\partial_{x+iy}\Phi^{(2)}\partial_{0-z}\Phi^{(1)} - \partial_{0-z}\Phi^{(2)}\partial_{x+iy}\Phi^{(1)}) \\
&= ig^2\sum_{j_1}\sum_{j_2}\sum_{j_3(\neq j_2)}\phi(j_1)\phi(j_2)\phi(j_3)(Q_{j_2}-Q_{j_3})^{-1}\left(X(j_1,j_2)+X(j_1,j_3)\right) \\
&+ ig^2\sum_{j_1}\sum_{j_2(\neq j_1)}\sum_{j_3}\phi(j_1)\phi(j_2)\phi(j_3)(Q_{j_1}-Q_{j_2})^{-1}\left(X(j_1,j_3)+X(j_2,j_3)\right) \\
&= ig^2\sum_{j_1}\sum_{j_2(\neq j_1)}\sum_{j_3(\neq j_2)}\phi(j_1)\phi(j_2)\phi(j_3)(Q_{j_2}-Q_{j_3})^{-1}\left(X(j_1,j_2)+X(j_1,j_3)\right) \\
&+ ig^2\sum_{j_1}\sum_{j_3(\neq j_1)}\phi(j_1)\phi(j_1)\phi(j_3)(Q_{j_1}-Q_{j_3})^{-1}\left(X(j_1,j_3)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \phi(j_1)\phi(j_2)\phi(j_3)(Q_{j_1}-Q_{j_2})^{-1}\left(X(j_1,j_3)+X(j_2,j_3)\right) \\
&+ \quad ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_2)\phi(j_2)(Q_{j_1}-Q_{j_2})^{-1}\left(X(j_1,j_2)\right) \\
&= \quad ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \phi(j_1)\phi(j_2)\phi(j_3)(Q_{j_1}-Q_{j_2})^{-1}(Q_{j_2}-Q_{j_3})^{-1} \\
&\qquad \times \left(X(j_1,j_2)(Q_{j_1}-Q_{j_2})+X(j_1,j_3)(Q_{j_1}-Q_{j_3})+X(j_2,j_3)(Q_{j_2}-Q_{j_3})\right) \\
&+ \quad ig^2 \sum_{j_1} \sum_{j_3(\neq j_1)} \phi(j_1)\phi(j_1)\phi(j_3)(Q_{j_1}-Q_{j_3})^{-1}\left(X(j_1,j_3)\right) \\
&+ \quad ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_2)\phi(j_2)(Q_{j_1}-Q_{j_2})^{-1}\left(X(j_1,j_2)\right).
\end{aligned}
$$

Note that

$$
(k_{j_1}+k_{j_2}+k_{j_3})^2 = X(j_1,j_2)(Q_{j_1}-Q_{j_2})+X(j_1,j_3)(Q_{j_1}-Q_{j_3})+X(j_2,j_3)(Q_{j_2}-Q_{j_3}),
$$

$$
(2k_{j_1}+k_{j_2})^2 = 2X(j_1,j_2)(Q_{j_1}-Q_{j_2}).
$$

Then the minimal solution of $\Phi^{(3)}(x)$ is given by

$$
\begin{aligned}
\Phi^{(3)}(x) \quad = \quad & -ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \phi(j_1)\phi(j_2)\phi(j_3)(Q_{j_1}-Q_{j_2})^{-1}(Q_{j_2}-Q_{j_3})^{-1} \\
& -ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_1)\phi(j_2)\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-2} \\
& -ig^2 \sum_{j_1} \sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_2)\phi(j_2)\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-2}.
\end{aligned} \tag{2.9}
$$

Similarly, we have the minimal solution for $\Phi^{(4)}(x)$:

$$
\begin{aligned}
& \Phi^{(4)}(x)/(-ig^3) \\
= \quad & \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \sum_{j_4(\neq j_3)} \phi(j_1)\phi(j_2)\phi(j_3)\phi(j_4)(Q_{j_1}-Q_{j_2})^{-1}(Q_{j_2}-Q_{j_3})^{-1}(Q_{j_3}-Q_{j_4})^{-1} \\
+ \quad & \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \phi(j_1)\phi(j_1)\phi(j_2)\phi(j_3)\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-2}(Q_{j_2}-Q_{j_3})^{-1} \\
+ \quad & \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \phi(j_1)\phi(j_2)\phi(j_2)\phi(j_3) \\
& \quad \times \left(\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-2}(Q_{j_2}-Q_{j_3})^{-1}+\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-1}(Q_{j_2}-Q_{j_3})^{-2}\right) \\
+ \quad & \sum_{j_1} \sum_{j_2(\neq j_1)} \sum_{j_3(\neq j_2)} \phi(j_1)\phi(j_2)\phi(j_3)\phi(j_3)\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-1}(Q_{j_2}-Q_{j_3})^{-2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j_1}\sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_1)\phi(j_1)\phi(j_2)\frac{1}{3!}(Q_{j_1}-Q_{j_2})^{-3} \\
&+ \sum_{j_1}\sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_1)\phi(j_2)\phi(j_2)\frac{1}{2}(Q_{j_1}-Q_{j_2})^{-3} \\
&+ \sum_{j_1}\sum_{j_2(\neq j_1)} \phi(j_1)\phi(j_2)\phi(j_2)\phi(j_2)\frac{1}{3!}(Q_{j_1}-Q_{j_2})^{-3}.
\end{aligned} \tag{2.10}
$$

Let us now consider the general case $\Phi^{(m)}(x)$. We define coefficients D as follows:

$$
\Phi^{(m)}(x) = -ig^{m-1}\sum_{j_1=1}^{n}\dots\sum_{j_m=1}^{n}\phi(j_1)\dots\phi(j_m)D(j_1,\dots,j_m). \tag{2.11}
$$

See (2.6). Then the recursion relation (2.1) can be transformed into the recursion relations for D:

$$
\begin{aligned}
&(k_{j_1}+\dots+k_{j_m})^2 D(j_1,\dots,j_m) \\
=& \sum_{l=1}^{m-1} D(j_1,\dots,j_l)D(j_{l+1},\dots,j_m)\left(\sum_{s=1}^{l}\sum_{t=l+1}^{m}X(j_s,j_t)\right).
\end{aligned} \tag{2.12}
$$

See (2.2). It is convenient to rearrange the sum (2.11) into the following form:

$$
\begin{aligned}
\Phi^{(m)} &= -ig^{m-1}\sum_{l=1}^{m}\sum_{j_1}\sum_{j_2(\neq j_1)}\dots\sum_{j_l(\neq j_{l-1})}\sum_{n_1\geq 1}\dots\sum_{n_l\geq 1} \\
&\quad\times\delta_{n_1+\dots+n_l,m}\phi(j_1)^{n_1}\phi(j_2)^{n_2}\dots\phi(j_l)^{n_l}D(1^{n_1}2^{n_2}\dots l^{n_l}).
\end{aligned} \tag{2.13}
$$

We introduce a notation:

$$
D(1^{n_1}2^{n_2}\dots l^{n_1}) = D(\overbrace{j_1,\dots,j_1}^{n_1},\overbrace{j_2,\dots,j_2}^{n_2},\dots,\overbrace{j_l,\dots,j_l}^{n_l}),
$$

$$
X_{st}=X(j_s,j_t),\qquad Q_{st}=Q_{j_s}-Q_{j_t}.
$$

The initial condition for $m=1$ and the minimality of the solution are summarized by

$$
D(1^{n_1})=\delta_{n_1,1}. \tag{2.14}
$$

Then the recursion relations (2.12) are given by

$$
\begin{aligned}
&\sum_{s<t} n_s n_t Q_{st}X_{st}D(1^{n_1}2^{n_2}\dots l^{n_l}) \\
=& \sum_{s<t}\sum_{p=1}^{n_s} p n_t X_{st} D(1^{n_1}2^{n_2}\dots(s-1)^{n_{s-1}}s^p)D(s^{n_s-p}(s+1)^{n_{s+1}}\dots l^{n_l})
\end{aligned}
$$

$$+ \sum_{s<t} \sum_{q=s+1}^{t-1} \sum_{p=1}^{n_q} n_s n_t X_{st} D(1^{n_1} 2^{n_2} \dots (q-1)^{n_{q-1}} q^p) D(q^{n_q - p} (q+1)^{n_{q+1}} \dots l^{n_l})$$

$$+ \sum_{s<t} \sum_{p=1}^{n_t - 1} n_s (n_t - p) X_{st} D(1^{n_1} 2^{n_2} \dots (t-1)^{n_{t-1}} t^p) D(t^{n_t - p} (t+1)^{n_{t+1}} \dots l^{n_l}). \qquad (2.15)$$

The solutions D do not contain the factor X_{st}. The above recursion relation is decomposed into the set of recursion relations by comparing the coefficients at X_{st} $(s < t)$. We have recursion relations:

$$\begin{aligned} & Q_{s,t} D(1^{n_1} 2^{n_2} \dots l^{n_l}) \\ = & \sum_{p=1}^{n_s} \frac{p}{n_s} D(1^{n_1} 2^{n_2} \dots (s-1)^{n_{s-1}} s^p) D(s^{n_s - p} (s+1)^{n_{s+1}} \dots l^{n_l}) \\ + & \sum_{q=s+1}^{t-1} \sum_{p=1}^{n_q} D(1^{n_1} 2^{n_2} \dots (q-1)^{n_{q-1}} q^p) D(q^{n_q - p} (q+1)^{n_{q+1}} \dots l^{n_l}) \\ + & \sum_{p=1}^{n_t - 1} \frac{(n_t - p)}{n_t} D(1^{n_1} 2^{n_2} \dots (t-1)^{n_{t-1}} t^p) D(t^{n_t - p} (t+1)^{n_{t+1}} \dots l^{n_l}). \end{aligned} \qquad (2.16)$$

Not all recursion relations (2.16) are independent. Note that

$$Q_{st} = \sum_{u=s+1}^{t} Q_{u-1,u}.$$

All Q_{st} can be expressed by $Q_{u-1,u}$ $(u = 2, \dots, l)$. The set of recursion relations (2.16) are generated from

$$\begin{aligned} & Q_{t-1,t} D(1^{n_1} 2^{n_2} \dots l^{n_l}) \\ = & \sum_{p=1}^{n_{t-1}} \frac{p}{n_{t-1}} D(1^{n_1} 2^{n_2} \dots (t-2)^{n_{t-2}} (t-1)^p) D((t-1)^{n_{t-1} - p} t^{n_t} \dots l^{n_l}) \\ + & \sum_{p=1}^{n_t - 1} \frac{(n_t - p)}{n_t} D(1^{n_1} 2^{n_2} \dots (t-1)^{n_{t-1}} t^p) D(t^{n_t - p} (t+1)^{n_{t+1}} \dots l^{n_l}), \end{aligned} \qquad (2.17)$$

for $t = 2, \dots, l$. But not all recursion relations (2.17) are needed to determine the coefficient D. With the initial condition (2.14), the recursion relation (2.17) for $t = 2$ is sufficient to obtain the answer. We must check that the solution satisfies the other recursion relation (2.17) for $t = 3, \dots, l$.

Combining eq.(2.14) and eq.(2.17) for $t = 2$, we have the recursion relation:

$$\begin{aligned} & D(1^{n_1} 2^{n_2} \dots l^{n_l}) \\ = & \frac{1}{n_1 n_2} Q_{12}^{-1} \left(n_2 D(1^{n_1 - 1} 2^{n_2} \dots l^{n_l}) + \sum_{p=1}^{n_2 - 1} n_1 (n_2 - p) D(1^{n_1} 2^p) D(2^{n_2 - p} 3^{n_3} \dots l^{n_l}) \right). \end{aligned} \qquad (2.18)$$

To solve this relation, we first determine the coefficient $D(1^{n_1}2^p)$. The simplest case is

$$D(12) = Q_{12}^{-1} D(1)D(2) = Q_{12}^{-1}.$$

For $n_2 > 1$, we have

$$D(12^{n_2}) = \frac{1}{n_2} Q_{12}^{-1} D(12^{n_2-1}) = \frac{1}{n_2!}(Q_{12}^{-1})^{n_2}. \tag{2.19}$$

Next we consider $D(1^{n_1}2^{n_2})$.

$$D(1^{n_1}2^{n_2}) = \frac{1}{n_1 n_2} Q_{12}^{-1} \left(n_2 D(1^{n_1-1}2^{n_2}) + n_1 D(1^{n_1}2^{n_2-1}) \right) \tag{2.20}$$

Let us define $C(n_1, n_2)$ by:

$$D(1^{n_1}2^{n_2}) = \frac{1}{n_1! n_2!}(Q_{12}^{-1})^{n_1+n_2-1} C(n_1, n_2). \tag{2.21}$$

Then the coefficient $C(n_1, n_2)$ satisfies

$$C(n_1, n_2) = C(n_1 - 1, n_2) + C(n_1, n_2 - 1). \tag{2.22}$$

With the initial conditions $C(1, n_2) = C(n_1, 1) = 1$, the solution is given by

$$C(n_1, n_2) = \frac{(n_1 + n_2 - 2)!}{(n_1 - 1)!(n_2 - 1)!}.$$

Therefore $D(1^{n_1}2^{n_2})$ is determined as

$$D(1^{n_1}2^{n_2}) = \frac{(n_1 + n_2 - 2)!}{n_1!(n_1 - 1)! n_2!(n_2 - 1)!}(Q_{12}^{-1})^{n_1+n_2-1}. \tag{2.23}$$

Let us insert this relation into eq.(2.18). Then we obtain

$$\begin{aligned} D(1^{n_1}2^{n_2}\dots l^{n_l}) &= \frac{1}{n_1} Q_{12}^{-1} D(1^{n_1-1}2^{n_2}\dots l^{n_l}) \\ &\quad + \sum_{p=1}^{n_2-1} \frac{(n_2 - p)}{n_1!\ p!\ n_2} C(n_1, p)(Q_{12}^{-1})^{n_1+p} D(2^{n_2-p}\dots l^{n_l}). \end{aligned} \tag{2.24}$$

Using this relation recursively, we have

$$\begin{aligned} D(1^{n_1}2^{n_2}\dots l^{n_l}) &= \frac{1}{n_1!}(Q_{12}^{-1})^{n_1} D(2^{n_2}\dots l^{n_l}) \\ &+ \sum_{p=1}^{n_2-1} \frac{(n_2 - p)}{n_1!\ p!\ n_2} \left(\sum_{s=1}^{n_1} C(n_1 + 1 - s, p) \right) (Q_{12}^{-1})^{n_1+p} D(2^{n_2-p}\dots l^{n_l}). \end{aligned} \tag{2.25}$$

With the help of the relation,

$$C(n_1, p+1) = \sum_{s=1}^{n_1} C(s, p), \tag{2.26}$$

we obtain the following recursion relations:

$$\begin{aligned} D(1^{n_1} 2^{n_2} \dots l^{n_l}) &= \sum_{p_2=0}^{n_2-1} \frac{(n_2 - p_2)\ C(n_1, p_2+1)}{n_1!\ p_2!\ n_2} (Q_{12}^{-1})^{n_1+p_2} D(2^{n_2-p_2} \dots l^{n_l}) \\ &= \sum_{p_2=1}^{n_2} \frac{p_2\ C(n_1, n_2+1-p_2)}{n_1!\ (n_2-p_2)!\ n_2} (Q_{12}^{-1})^{n_1+n_2-p_2} D(2^{p_2} \dots l^{n_l}). \end{aligned} \tag{2.27}$$

Here we replaced p_2 with $n_2 - p_2$.

Then, solutions of the recursion relations for D are finally given by

$$\begin{aligned} & D(1^{n_1} 2^{n_2} \dots l^{n_l}) \\ = & \sum_{p_2=1}^{n_2} \sum_{p_3=1}^{n_3} \cdots \sum_{p_{l-1}=1}^{n_{l-1}} \prod_{s=1}^{l} (n_s)^{-1} \prod_{t=2}^{l} \left(\frac{(p_{t-1} + n_t - p_t - 1)!}{((p_{t-1}-1)!(n_t - p_t)!)^2} (Q_{t-1,t}^{-1})^{p_{t-1}+n_t-p_t} \right) \end{aligned} \tag{2.28}$$

with the convention $p_1 = n_1$ and $p_l = 1$. We can see that this solution (2.28) indeed satisfies recursion relations (2.17). Combination of formulas (1.4), (2.13) and (2.28) gives the perturbative solution of self-dual equation (1.3).

Some explicit examples of D are given by:

$$D(12 \dots m) = Q_{12}^{-1} Q_{23}^{-1} \dots Q_{m-1,m}^{-1}, \tag{2.29}$$

$$D(1^{n_1} 2 \dots l) = \frac{1}{n_1!} (Q_{12}^{-1})^{n_1} Q_{23}^{-1} \dots Q_{l-1,l}^{-1}. \tag{2.30}$$

3 Scattering Operator

Let us simplify the expression for the solution:

$$\begin{aligned} \Phi(x) &= \sum_{m=1}^{\infty} \Phi^{(m)}(x) \\ &= -i \sum_{m=1}^{\infty} g^{m-1} \sum_{l=1}^{m} \sum_{j_1} \sum_{j_2(\neq j_1)} \cdots \sum_{j_l(\neq j_{l-1})} \sum_{n_1=1}^{\infty} \cdots \sum_{n_l=1}^{\infty} \\ &\qquad \times \delta_{n_1+\dots+n_l, m} \phi(j_1)^{n_1} \dots \phi(j_l)^{n_l} D(1^{n_1} \dots l^{n_l}) \\ &= -i \sum_{l=1}^{\infty} \sum_{\{j_i \neq j_{i+1}\}} \sum_{\{n_i \geq 1\}} g^{n_1+\dots+n_l-1} \phi(j_1)^{n_1} \dots \phi(j_l)^{n_l} D(1^{n_1} \dots l^{n_l}). \end{aligned} \tag{3.1}$$

See (2.13) and (2.28). To obtain the last expression, we first exchange the order of summation with respect to m and l. Then we use $\delta_{n_1+\ldots+n_l,m}$ in order to perform summation with respect to m.

The coefficients D depend on the momenta by means of $Q^{-1}_{i-1,i} = (Q_{j_{i-1}} - Q_{j_i})^{-1}$:

$$D(1^{n_1}\ldots l^{n_l}) = D(1^{n_1}\ldots l^{n_l}; Q^{-1}_{12}, \ldots, Q^{-1}_{l-1,l}).$$

The factor $Q^{-1}_{i-1,i}$ diverges for $j_{i-1} = j_i$. To regularize this divergence, we replace $Q^{-1}_{i-1,i}$ in D with $P_{i-1,i} = P(j_{i-1}, j_i)$. See eq.(1.12). Then the solution of the self-dual equation can be written as

$$\Phi(x) = \sum_{l=1}^{\infty} \tilde{\Phi}^{(l)}(x). \tag{3.2}$$

This is not the expansion in the coupling constant. $\tilde{\Phi}^{(l)}$ is not homogeneous in g. Here

$$\tilde{\Phi}^{(l)}(x) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_l=1}^{n} \tilde{\Phi}^{(l)}(j_1, \ldots, j_l), \tag{3.3}$$

and

$$\begin{aligned}\tilde{\Phi}^{(l)}(j_1,\ldots,j_l) &= -i\sum_{n_1=0}^{\infty}\cdots\sum_{n_l=0}^{\infty} g^{n_1+\ldots+n_l+l-1} \\ &\quad\times\phi(j_1)^{n_1+1}\ldots\phi(j_l)^{n_l+1} D(1^{n_1+1}\ldots l^{n_l+1}; P_{12}, \ldots, P_{l-1,l}).\end{aligned} \tag{3.4}$$

Especially for $l = 1$,

$$\tilde{\Phi}^{(1)}(j_1) = -i\phi(j_1) = -iT^{a_{j_1}} e^{-ik_{j_1}x} f(k_j). \tag{3.5}$$

We consider $\tilde{\Phi}^{(2)}(j_1, j_2)$.

$$\tilde{\Phi}^{(2)}(j_1,j_2) = -ig^{-1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(g\phi(j_1))^{n_1+1}(g\phi(j_2))^{n_2+1}\frac{(n_1+n_2)!}{(n_1+1)!n_1!(n_2+1)!n_2!}(P_{12})^{n_1+n_2+1}. \tag{3.6}$$

Note that $\phi(j_1)$ takes a value in the Lie algebra of the gauge group. The order of ϕ's needs to be considered.

We shall use the integral representation for the Gamma function

$$m! = \int_0^{\infty} ds\ e^{-s}\ \ s^m, \tag{3.7}$$

to represent $(n_1+n_2)!$ in (3.6). We shall obtain

$$\tilde{\Phi}^{(2)}(j_1,j_2) = -ig^{-1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int_0^\infty ds\ e^{-s}\frac{s^{n_1+n_2}(g\phi(j_1))^{n_1+1}(g\phi(j_2))^{n_2+1}(P_{12})^{n_1+n_2+1}}{(n_1+1)!n_1!(n_2+1)!n_2!}. \tag{3.8}$$

Now the summation in n_1 and n_2 factorizes.

The sum in n_1 or n_2 in the above equation can be expressed in terms of:

$$V_1(a) = \sum_{n=0}^{\infty}\frac{1}{(n+1)!n!}a^n = \int_0^1 d\alpha\ V(\alpha a). \tag{3.9}$$

Here $V(a)$ is given by eq.(1.10). Note that $V_1(a)$ has several representations:

$$V_1(a) = \oint\frac{dt}{2\pi i}e^{1/t+at} = I_1(2\sqrt{a})/\sqrt{a} = \frac{d}{da}V(a). \tag{3.10}$$

Here I_1 is the modified Bessel function of the first kind. The integration contour is a circle in the complex plane around zero in positive direction.

Now the solution $\tilde{\Phi}^{(2)}(j_1,j_2)$ can be expressed as

$$\tilde{\Phi}^{(2)}(j_1,j_2) = -i\int_0^1 d\alpha_1\int_0^1 d\alpha_2 T(\alpha_1,\alpha_2;j_1,j_2)\phi(j_2), \tag{3.11}$$

where $T(\alpha_1,\alpha_2;j_1,j_2)$ is defined by

$$\begin{aligned} &T(\alpha_1,\alpha_2;j_1,j_2)\\ =\ &g\phi(j_1)P(j_1,j_2)\int_0^\infty ds\ e^{-s}V\Big(s\alpha_1 g\phi(j_1)P(j_1,j_2)\Big)V\Big(s\alpha_2 g\phi(j_2)P(j_1,j_2)\Big). \end{aligned} \tag{3.12}$$

Then $\tilde{\Phi}^{(2)}(x)$ is given by

$$\tilde{\Phi}^{(2)}(x) = -i\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\int_0^1 d\alpha_1\int_0^1 d\alpha_2 T(\alpha_1,\alpha_2;j_1,j_2)\phi(j_2). \tag{3.13}$$

The integration variable α was introduced by formula (3.9).

Using a similar technique, we can write the coefficient D (2.28) as follows:

$$\begin{aligned} &D(1^{n_1+1}2^{n_2+1}\dots l^{n_l+1})\\ =\ &\sum_{p_2=0}^{n_2}\dots\sum_{p_{l-1}=0}^{n_{l-1}}\prod_{t=2}^{l}\int_0^\infty ds_t\ e^{-s_t}P_{t-1,t}\\ &\times\frac{(s_2P_{12})^{n_1}}{(n_1+1)!n_1!}\prod_{t=2}^{l-1}\frac{(s_tP_{t-1,t})^{n_t-p_t}(s_{t+1}P_{t,t+1})^{p_t}}{(n_t+1)(p_t!(n_t-p_t)!)^2}\frac{(s_lP_{l-1,l})^{n_l}}{(n_l+1)!n_l!}. \end{aligned} \tag{3.14}$$

We replaced p_t with p_t+1 and represented $(p_{t-1}+n_t-p_t)!$ by s_t-integration. In order to calculate $\tilde{\Phi}^{(l)}$, we need to substitute this expression for D into the sum (3.4). Then we have

$$\begin{aligned}
&\tilde{\Phi}^{(l)}(j_1,\dots,j_l) \\
=\ & -i\sum_{n_1=0}^{\infty}\dots\sum_{n_l=0}^{\infty}\sum_{p_2=0}^{n_2}\dots\sum_{p_{l-1}=0}^{n_{l-1}}\prod_{t=2}^{l}\int_0^\infty ds_t\ e^{-s_t}g\phi(j_1)P_{12}\frac{(s_2 g\phi(j_1)P_{12})^{n_1}}{(n_1+1)!n_1!} \\
&\times\prod_{t=2}^{l-1}\frac{(s_t g\phi(j_t)P_{t-1,t})^{n_t-p_t}g\phi(j_t)P_{t,t+1}(s_{t+1}g\phi(j_t)P_{t,t+1})^{p_t}}{(n_t+1)(p_t!(n_t-p_t)!)^2}\frac{(s_l g\phi(j_l)P_{l-1,l})^{n_l}}{(n_l+1)!n_l!}\phi(j_l).
\end{aligned} \tag{3.15}$$

Using formula (3.9), we introduce integration over α_1 and α_l and perform the summation in n_1 and n_l:

$$\begin{aligned}
&\tilde{\Phi}^{(l)}(j_1,\dots,j_l) \\
=\ & -i\sum_{n_2=0}^{\infty}\dots\sum_{n_{l-1}=0}^{\infty}\sum_{p_2=0}^{n_2}\dots\sum_{p_{l-1}=0}^{n_{l-1}}\prod_{t=2}^{l}\int_0^\infty ds_t\ e^{-s_t}\int_0^1 d\alpha_1\int_0^1 d\alpha_l g\phi(j_1)P_{12}V(\alpha_1 s_2 g\phi(j_1)P_{12}) \\
&\times\prod_{t=2}^{l-1}\frac{(s_t g\phi(j_t)P_{t-1,t})^{n_t-p_t}g\phi(j_t)P_{t,t+1}(s_{t+1}g\phi(j_t)P_{t,t+1})^{p_t}}{(n_t+1)(p_t!(n_t-p_t)!)^2}V(\alpha_l s_l g\phi(j_l)P_{l-1,l})\phi(j_l).
\end{aligned} \tag{3.16}$$

In order to calculate summation in n_t and p_t for $j=2,\dots,l-1$, we need to calculate the sum:

$$\sum_{n_t=0}^{\infty}\sum_{p_t=0}^{n_t}\frac{(s_t g\phi(j_t)P_{t-1,t})^{n_t-p_t}g\phi(j_t)P_{t,t+1}(s_{t+1}g\phi(j_t)P_{t,t+1})^{p_t}}{(n_t+1)(p_t!(n_t-p_t)!)^2}. \tag{3.17}$$

We first exchange the order of summation then replace n_t with n_t+p_t:

$$\begin{aligned}
&\sum_{p_t=0}^{\infty}\sum_{n_t=p_t}^{\infty}\frac{(s_t g\phi(j_t)P_{t-1,t})^{n_t-p_t}g\phi(j_t)P_{t,t+1}(s_{t+1}g\phi(j_t)P_{t,t+1})^{p_t}}{(n_t+1)(p_t!(n_t-p_t)!)^2} \\
=\ &\sum_{p_t=0}^{\infty}\sum_{n_t=0}^{\infty}\frac{(s_t g\phi(j_t)P_{t-1,t})^{n_t}g\phi(j_t)P_{t,t+1}(s_{t+1}g\phi(j_t)P_{t,t+1})^{p_t}}{(n_t+p_t+1)(p_t!n_t!)^2}.
\end{aligned} \tag{3.18}$$

Let us represent $(n_t+p_t+1)^{-1}$ by

$$\frac{1}{n_t+p_t+1}=\int_0^1 d\alpha_t\ \alpha_t^{n_t+p_t}. \tag{3.19}$$

This formula introduces α_t-integration. The sum in (3.17) can be written as

$$\int_0^1 d\alpha_t\left(\sum_{n_t=0}^{\infty}\frac{(\alpha_t s_t g\phi(j_t)P_{t-1,t})^{n_t}}{(n_t!)^2}\right)g\phi(j_t)P_{t,t+1}\left(\sum_{p_t=0}^{\infty}\frac{(\alpha_t s_{t+1}g\phi(j_t)P_{t,t+1})^{p_t}}{(p_t!)^2}\right). \tag{3.20}$$

By means of (1.10) the sum (3.17) is equivalent to

$$\int_0^1 d\alpha_t V(\alpha_t s_t g\phi(j_t)P_{t-1,t})g\phi(j_t)P_{t,t+1}V(\alpha_t s_{t+1} g\phi(j_t)P_{t,t+1}).$$

Then, we have

$$\begin{aligned}
&\tilde{\Phi}^{(l)}(j_1, j_2, \ldots, j_l) \\
= \; & -i\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 d\alpha_l \prod_{t=2}^{l} \int_0^\infty ds_t e^{-s_t} \; g\phi(j_1)P_{12}V(\alpha_1 s_2 g\phi(j_1)P_{12}) \\
& \times \prod_{t=2}^{l-1}\Big(V(\alpha_t s_t g\phi(j_t)P_{t-1,t})g\phi(j_t)P_{t,t+1}V(\alpha_t s_{t+1} g\phi(j_t)P_{t,t+1})\Big)V(\alpha_l s_l g\phi(j_l)P_{l-1,l})\phi(j_l) \\
= \; & -i\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 d\alpha_l \\
& \times \prod_{t=2}^{l}\Big(\int_0^\infty ds_t e^{-s_t} g\phi(j_{t-1})P_{t-1,t}V(\alpha_{t-1}s_t g\phi(j_{t-1})P_{t-1,t})V(\alpha_t s_t g\phi(j_t)P_{t-1,t})\Big)\phi(j_l).
\end{aligned}$$

Recalling the definition of T (3.12), we have

$$\begin{aligned}
\tilde{\Phi}^{(l)}(j_1, j_2, \ldots, j_l) \;=\; & -i\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 d\alpha_l \\
& \times T(\alpha_1, \alpha_2; j_1, j_2)T(\alpha_2, \alpha_3; j_2, j_3)\ldots T(\alpha_{l-1}, \alpha_l; j_{l-1}, j_l)\phi(j_l). \quad (3.21)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tilde{\Phi}^{(l)}(x) \;=\; & -i\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\ldots\sum_{j_l=1}^{n}\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 d\alpha_l \\
& \times T(\alpha_1, \alpha_2; j_1, j_2)T(\alpha_2, \alpha_3; j_2, j_3)\ldots T(\alpha_{l-1}, \alpha_l; j_{l-1}, j_l)\phi(j_l). \qquad (3.22)
\end{aligned}$$

See (3.3). Using the 'scattering operator', we obtain a simple representation for $\tilde{\Phi}^{(l)}$,

$$\tilde{\Phi}^{(l)}(x) = -i\vec{\phi}_0 \cdot (\hat{T})^{l-1}\vec{\phi}. \qquad (3.23)$$

See (1.14). Thus, we proved our main formula (1.15):

$$\Phi(x) = -i\vec{\phi}_0 \cdot \left(1 + \sum_{l=1}^{\infty}(\hat{T})^l\right)\vec{\phi} = -i\vec{\phi}_0 \cdot \left(\frac{1}{1-\hat{T}}\right)\vec{\phi}. \qquad (3.24)$$

4 Summary

Introduction of an auxiliary linear space is typical for the solution of two-dimensional completely integrable differential equations [11, 12]. In this sense, our solution is represented similar to solutions of two-dimensional classical completely integrable differential equations.

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